An Elementary Approach to a Model Problem of Lagerstrom

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Abstract

The equation studied is $u'' + \frac{n-1}{r}u' + \varepsilon u \ u' + ku'^2 = 0$, with boundary conditions $u(1) = 0$, $u(\infty) = 1$. This model equation has been studied by many authors since it was introduced in the 1950s by P. A. Lagerstrom. We use an elementary approach to show that there is an infinite series solution which is uniformly convergent on $1 \leq r < \infty$. The first few terms are easily derived, from which one quickly deduces the inner and outer asymptotic expansions, with no matching procedure or a priori assumptions about the nature of the expansion. We also give a short and elementary existence and uniqueness proof which covers all $\varepsilon > 0$, $k \geq 0$, and $n \geq 1$.

1 Introduction

The main problem is to investigate the asymptotics as $\varepsilon \to 0$ of the boundary value problem

$$u'' + \frac{n-1}{r}u' + \varepsilon uu' + ku'^2 = 0 \tag{1}$$

with

$$u(1) = 0, u(\infty) = 1. \tag{2}$$

We consider the cases $k = 0$ and $k = 1$. Our interest in these problems, originally due to Lagerstrom in the 1950s [6],[7], was stimulated by two recent papers by Popovic and Szmolyan [10],[11], who adopt a geometric approach to the problem when $k = 0$, and there are many papers which use methods of matched asymptotics or multiple scales, with varying degrees of rigor. We will review some of this work below. The point of this paper is to give a completely rigorous and relatively short

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answer to the problem without making any appeal either to geometric methods or to matched asymptotics. We can express the solution as an infinite series, uniformly convergent for all values of the independent variable. From this series we obtain the inner and outer asymptotic expansions with no a priori assumption about the nature of these expansions. An important, and as far as we know, original, feature is that there is no "matching".

Lagerstrom came up with these problems as models of viscous incompressible ($k = 0$) and compressible ($k = 1$) flow, so much of his work centered on $n = 2$ or 3, but he also discussed general $n \geq 1$ [8]. The infinite series we develop can be obtained for any real number n. What n controls is the rate of convergence of the series.

For $\varepsilon = k = 0$, there is an obvious distinction between $n > 2$ and $n \leq 2$. If $n > 2$, then the problem $(1) - (2)$ has the unique solution

$$u = 1 - \frac{1}{r^{n-2}}, \tag{3}$$

so that the solution with ε small is presumably some sort of perturbation of this. If $n \leq 2$ then there is no such solution. A consequence is that the convergence as $\varepsilon \to 0$ is more subtle when $n \leq 2$ then when $n > 2$. Our analysis will show that there is little prospect of discussing the behavior for small ε if $n < 2$, but fortunately we can handle all $n \geq 2$. Although it has been thought that finding the asymptotics when $k = 1$ is considerably more difficult than when $k = 0$, [3], we will show that our technique covers each case with comparable effort.

Our methods are not restricted to Lagerstrom's problems $(1) - (2)$. In subsequent work (in preparation), we will show that there is a general method which can yield similar results for a class of singularly perturbed boundary value problems.

We start in section 2 by showing that each of these problems has one and only one solution, for any $n \geq 1$ and any $\varepsilon > 0$. This is based on a simple shooting argument plus a comparison principle. These results have been obtained before, but our proof is quite short. In the subsequent sections we develop the integral equation referred to above, and show how it leads with relative ease to the inner and outer expansions. These expansions go back to Lagerstrom and Kaplun, with rigorous justification of some of the features to be found in [1] or [11], for example. We find the exposition in Hinch's book [3] particularly clear (though nonrigorous),

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and make that our point of comparison in checking that we get the same expansions as were found previously.[1]

2   Existence and uniqueness

As far as we know, the first existence proof was by Hsiao [4], who only considered $n = 1$ and sufficiently small $\varepsilon > 0$. Subsequently Tam gave what seems to be the first proof valid for all $\varepsilon > 0$ and $k \geq 0$, [14]. Subsequent proofs by MacGillivray [9], Cohen, Lagerstrom and Fokas [1], Hunter, Tajdari, and Boyer [5], each of which covers all $\varepsilon > 0$, and by several other authors, e.g. [12],[10], for restricted ranges of ε, add to the variety of techniques which have been shown to work. Uniqueness is proved in [5] (for $k = 0$) by use of a contraction mapping theorem, and in [1] by essentially a comparison method. The goal of [10] is not to give a short proof, but to illustrate the application of geometric perturbation theory to a much studied problem in matched asymptotic expansions. The proofs we give of existence and uniqueness are considerably shorter than the others we have seen.

Theorem 1   *There exists a unique solution to the problem* $(1) - (2)$ *for any* $k \geq 0$, $\varepsilon > 0$, *and* $n \geq 1$.

Proof. Like some others, starting with [14], we prove existence using a shooting method, by considering the initial value problem

$$u'' + \frac{n-1}{r}u' + \varepsilon u u' + k u'^2 = 0 \tag{4}$$

$$u(1) = 0, \ u'(1) = c, \tag{5}$$

for each $c > 0$. Since $u' = 0$ implies that $u'' = 0$ and u is constant, any solution to this problem is positive and increasing. As was observed in [14],

$$u'' + \varepsilon u u' \leq 0,$$

and so from (5),

$$u' + \frac{1}{2}\varepsilon u^2 \leq c.$$

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[1]We thank the referees for some very helpful comments. In particular, they called our attention to earlier proofs of the existence and uniqueness results, in some cases by techniques similar to ours.

In particular, since $u' \geq 0$,

$$u \leq \sqrt{\frac{2c}{\varepsilon}}, \tag{6}$$

so the solution exists, and satisfies this bound, on $[1, \infty)$. Therefore, $\lim_{r \to \infty} u(r)$ exists. Writing the equation in the form

$$\left(r^{n-1} u'\right)' + \left(\varepsilon u + k u'\right)\left(r^{n-1} u'\right) = 0, \tag{7}$$

and integrating twice, gives

$$r^{n-1} u'(r) = c e^{-ku(r) - \varepsilon \int_1^r u(s)\,ds}, \tag{8}$$

$$u(r) = \int_1^r \frac{c}{s^{n-1}} e^{-ku - \int_1^s \varepsilon u\,dt}\,ds. \tag{9}$$

If $u(2) < 1$, then since u is increasing, (9) implies that

$$u(2) > p(c) = \int_1^2 \frac{c}{s^{n-1}} e^{-\varepsilon - k}\,ds.$$

From this, and (6), we see that there are c_1 and c_2, with $0 < c_1 < c_2$, such that if $c = c_1$ then $u(\infty) < 1$, while if $c = c_2$, then $u(\infty) > u(2) \geq 1$. Further, from (8) for any $r > R > 2$,

$$u(r) = u(R) + c \int_R^r \frac{1}{s^{n-1}} e^{-ku - \int_1^s \varepsilon u\,dt}\,ds.$$

If $n > 2$ the second term is bounded above by $\frac{c}{(n-2)R^{n-2}}$, while if $1 \leq n \leq 2$ and $R \geq 2$, it is bounded by $c \int_R^\infty e^{-\varepsilon(s-2)p(c)}\,ds$. Hence, this term tends to zero as $R \to \infty$, uniformly for $r \geq R$, $c_1 \leq c \leq c_2$. Since $u(R)$ is a continuous function of c, for any R, it follows that $u(\infty)$ is also continuous in c, and so there is a c with $u(\infty) = 1$, giving a solution to $(1) - (2)$.

For uniqueness, suppose that there are two solutions of $(1) - (2)$, say u_1 and u_2, with $u_1'(1) > u_2'(1) > 0$. Then $u_1 > u_2$ on some maximal interval, say $(1, X)$ where $X \leq \infty$. For the same initial conditions, if $\varepsilon = k = 0$ then direct integration shows that $u_1 > u_2$ on $(1, \infty)$, and moreover, $u_1(\infty) > u_2(\infty)$. We then raise ε and k, looking for a pair (ε_1, k_1) such that $u_1(X) = u_2(X)$ for some $X \leq \infty$, and if $0 \leq \varepsilon < \varepsilon_1$ or $0 \leq k < k_1$, no such X exists. Hence, at (ε_1, k_1), $u_1 \geq u_2$ on $[0, \infty)$. If, at (ε_1, k_1), $X < \infty$, then u_1 and u_2 must be tangent at X, since $u_1 - u_2$ has a minimum there, contradicting the uniqueness of initial value problems for (1). Hence, $X = \infty$, and $u_1 > u_2$ on $(1, \infty)$.

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Observe from (7) that if $u_1'(r) = u_2'(r)$ for some r, then $(r^{n-1}(u_1' - u_2'))' < 0$, since $u_1 > u_2$, so that there cannot be oscillations in $u_1' - u_2'$. Hence, $u_1(\infty) = u_2(\infty)$ implies that there is an R with $u_1'(R) = u_2'(R)$ and $u_1' < u_2'$ on (R, ∞). Integrating (7), and recalling that $u_1(\infty) = u_2(\infty)$, gives

$$r^{n-1}(u_1' - u_2')\,|_R^\infty = \frac{1}{2}\varepsilon R^{n-1}\left(u_1^2 - u_2^2\right)|_R$$
$$+ \frac{1}{2}\varepsilon(n-1)\int_R^\infty s^{n-2}\left(u_1^2 - u_2^2\right)ds - k\int_R^\infty s^{n-1}\left(u_1'^2 - u_2'^2\right)ds.$$

The left hand side is zero, and all the terms on the right are positive, giving the necessary contradiction. ∎

Remark 1 *The existence theorem in [10] has one added part. It is shown there that as $\varepsilon \to 0$, the solution tends to a so-called "singular" solution obtained by taking a formal limit as $\varepsilon \to 0$. See [10] for details. This limit result follows from our rigorous asymptotic expansions given below.*

Remark 2 *There would seem to be no difficulty in extending the existence proof even to $n < 1$, but the uniqueness proof does use essentially the fact that $n \geq 1$.*

3   The infinite series (with $k = 0$, $n \geq 2$)

Starting again with (1), and $u(1) = 0$, we first consider the case $k = 0$, and obtain

$$r^{n-1}u' = Be^{-\varepsilon \int_1^r u(t)dt} \tag{10}$$

for some constant B. Since $u(\infty) = 1$, (10) implies that $u'(r)$ is exponentially small as $r \to \infty$. Hence we can rewrite (10) as

$$r^{n-1}u' = Ce^{-\varepsilon r - \varepsilon \int_\infty^r (u-1)dt},$$

so that

$$u - 1 = C\int_\infty^r \frac{1}{t^{n-1}}e^{-\varepsilon t - \varepsilon \int_\infty^t (u-1)ds}dt.$$

Setting $\varepsilon r = \rho$, $\varepsilon t = \tau$, and $\varepsilon s = \sigma$, we obtain

$$u(\rho) - 1 = C\varepsilon^{n-2}\int_\infty^\rho \frac{1}{\tau^{n-1}}e^{-\tau}e^{-\int_\infty^\tau (u(\sigma)-1)d\sigma}d\tau, \tag{11}$$

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where we use the arguments ρ and σ to indicate that we mean the rescaled version of u. Here C is a constant satisfying

$$-1 = C\varepsilon^{n-2} \int_\infty^\varepsilon \frac{1}{\tau^{n-1}} e^{-\tau} e^{-\int_\infty^\tau (u-1)d\sigma} d\tau. \tag{12}$$

Since for each ε there is a unique solution, this determines a unique C, dependent on ε.

We now consider the integral

$$\int_\tau^\infty (1 - u(\sigma)) \, d\sigma, \tag{13}$$

which appears in the exponent in (12). This integral has been seen to converge for each ε, but we need a bit more, namely, that it is bounded uniformly in $\varepsilon \leq \tau < \infty$ as $\varepsilon \to 0$. To see this, we note that as a function of σ, u satisfies

$$\frac{d^2 u}{d\sigma^2} + \frac{n-1}{\sigma} \frac{du}{d\sigma} + u \frac{du}{d\sigma} = 0$$
$$u = 0 \text{ when } \sigma = \varepsilon, \ u(\infty) = 1.$$

Denoting the unique solution by $u_\varepsilon(\sigma)$, we claim that if $0 < \varepsilon_1 < \varepsilon_2$, then $u_{\varepsilon_1} > u_{\varepsilon_2}$ for $\varepsilon_2 \leq \sigma < \infty$. If this is false, then ε_1 and ε_2 can be chosen so that $u_{\varepsilon_1}(\sigma_0) = u_{\varepsilon_2}(\sigma_0)$ for some $\sigma_0 \geq \varepsilon_2$. But then, the problem

$$\frac{d^2 u}{d\sigma^2} + \frac{n-1}{\sigma} \frac{du}{d\sigma} + u \frac{du}{d\sigma} = 0$$
$$u(\sigma_0) = u_{\varepsilon_1}(\sigma_0), \ u(\infty) = 1$$

has two solutions, contradicting our earlier uniqueness proof.

A consequence of this is that $\int_{\varepsilon_2}^\infty (1 - u_{\varepsilon_1}(\sigma)) \, d\sigma < \int_{\varepsilon_2}^\infty (1 - u_{\varepsilon_2}(\sigma)) \, d\sigma$, which implies that the integral in the exponent in (12), including the minus sign in front, is bounded below independently of $\tau \geq \varepsilon$, and of ε. We then see that the τ-integral in (12) approaches $-\infty$ as $\varepsilon \to 0$, and hence, that

$$\lim_{\varepsilon \to 0^+} C\varepsilon^{n-2} = 0.$$

Since $\int_\infty^\tau (u - 1) \, d\sigma > 0$, it follows from (11) that if

$$E_{n-1}(\rho) = \int_\rho^\infty \frac{1}{\tau^{n-1}} e^{-\tau} d\tau,$$

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then
$$|u(\rho) - 1| < C\varepsilon^{n-2}E_{n-1}(\rho).\tag{14}$$

For purposes of future estimates, we make the obvious remark that

$$E_{n-1}(\rho) = \begin{cases} O(\rho^{2-n}) & \text{as } \rho \to 0 \text{ if } n > 2 \\ O(\log \rho) & \text{as } \rho \to 0 \text{ if } n = 2 \\ O(\rho^{1-n}e^{-\rho}) & \text{as } \rho \to \infty. \end{cases}\tag{15}$$

Hence if $n > 2$ there is a constant K such that

$$E_{n-1}(\rho) \leq K \min\left(\rho^{2-n}, \rho^{1-n}e^{-\rho}\right).\tag{16}$$

The method now is to work from (11). As observed before, since $u'(r)$ is exponentially small as $r \to \infty$, the integral term $\int_\rho^\infty (u-1)\, d\sigma$ converges. Hence, for given $\varepsilon > 0$ and $\rho_0 > 0$, and any $\rho \geq \rho_0$,

$$u(\rho) - 1 = C\varepsilon^{n-2}\int_\infty^\rho \frac{1}{\tau^{n-1}}e^{-\tau}\left\{1 - \int_\infty^\tau (u-1)\, d\sigma + \frac{1}{2}\left(\int_\infty^\tau (u-1)\, d\sigma\right)^2 - \cdots\right\} d\tau,\tag{17}$$

where the series in the integrand converges uniformly for $\rho_0 \leq \tau < \infty$.

In fact, we will need to use this series for all $\rho \geq \varepsilon$. Thus we need to check its convergence in this interval. This follows from (14) and (15), which imply that for any $\rho \geq \varepsilon$, if $n \geq 2$, then

$$\left|\int_\rho^\infty (u(s) - 1)\, ds\right| < C\varepsilon^{n-2}\int_\varepsilon^\infty E_{n-1}(s)\, ds\tag{18}$$

and

$$\varepsilon^{n-2}\int_\varepsilon^\infty E_{n-1}(s)\, ds = \begin{cases} o(1) & \text{as } \varepsilon \to 0 \text{ if } n > 2 \\ O(1) & \text{as } \varepsilon \to 0 \text{ if } n = 2 \end{cases}.$$

Hence for $n > 2$ and any C, the series in the integrand of (17) converges uniformly on $[\varepsilon, \infty)$.

Now set

$$\Phi = C\varepsilon^{n-2}\int_\varepsilon^\infty E_{n-1}(s)\, ds.$$

We note that, if $n > 2$, then $\Phi \to 0$ as $\varepsilon \to 0$, while if $n = 2$, then $\Phi \to 0$ as $C \to 0$.

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We proceed to solve (17) by iteration. Thus, the first approximation is, from (16),

$$u\left(\rho\right) - 1 = C\varepsilon^{n-2} \int_{\infty}^{\rho} \frac{1}{\tau^{n-1}} e^{-\tau} d\tau + O\left(\Phi^2\right),$$

and we obtain the second approximation by substituting this back in (17). Repeating this, we reach

$$
\begin{aligned}
u - 1 =& -C\varepsilon^{n-2} E_{n-1} + \left(C\varepsilon^{n-2}\right)^2 \int_{\infty}^{\rho} \frac{1}{\tau^{n-1}} e^{-\tau} \left(\int_{\infty}^{\tau} E_{n-1} d\sigma\right) d\tau \\
& + \tfrac{1}{2} \left(C\varepsilon^{n-2}\right)^3 \int_{\infty}^{\rho} \frac{1}{\tau^{n-1}} e^{-\tau} \left(\int_{\infty}^{\tau} E_{n-1} d\sigma\right)^2 d\tau \\
& - \left(C\varepsilon^{n-2}\right)^3 \int_{\infty}^{\rho} \frac{1}{\tau^{n-1}} e^{-\tau} \int_{\infty}^{\tau} \left\{\int_{\infty}^{\sigma} \frac{1}{s^{n-1}} e^{-s} \left(\int_{\infty}^{s} E_{n-1} dt\right) ds\right\} d\sigma d\tau + O\left(\Phi^4\right),
\end{aligned}
\tag{19}
$$

as $\Phi \to 0$.

To obtain C, we need to be able to evaluate each of these terms for small ρ (in particular, for $\rho = \varepsilon$), and this is a matter of integration by parts. Thus, for non-integral n,

$$
\begin{aligned}
E_{n-1}\left(\rho\right) = \int_{\rho}^{\infty} \frac{e^{-\tau}}{\tau^{n-1}} d\tau &= -\frac{\rho^{2-n}}{2-n} e^{-\rho} + \frac{1}{2-n} \int_{\rho}^{\infty} \frac{e^{-\tau}}{\tau^{n-2}} d\tau \\
&= -\frac{\rho^{2-n}}{2-n} e^{-\rho} + \frac{1}{2-n} E_{n-2},
\end{aligned}
\tag{20}
$$

and this can be repeated to give E_{n-1} as a sum of terms of the form $c_k \rho^k e^{-\rho}$ and E_{n-p}, until $0 < n - p < 1$. Then

$$
\begin{aligned}
E_{n-p} &= \int_{0}^{\infty} \frac{e^{-\tau}}{\tau^{n-p}} d\tau - \int_{0}^{\rho} \frac{e^{-\tau}}{\tau^{n-p}} d\tau \\
&= \Gamma\left(p + 1 - n\right) - \int_{0}^{\rho} \frac{e^{-\tau}}{\tau^{n-p}} d\tau
\end{aligned}
$$

and we can then continue to integrate by parts as far as we like. (If n is an integer, we will reach $\int_{\rho}^{\infty} \frac{e^{-\tau}}{\tau} d\tau$, which introduces a logarithm.)

Thus $E_{n-1}\left(\rho\right)$ can be expressed as a sum of terms of the form $c_k \rho^k e^{-\rho}$, and so obviously the same is true of E_{n-1}^2, with $e^{-2\rho}$ in place of $e^{-\rho}$. Also,

$$
\begin{aligned}
\int_{\infty}^{\rho} E_{n-1}\left(\tau\right) d\tau &= \int_{\infty}^{\rho} \left(\int_{\tau}^{\infty} \frac{e^{-\sigma}}{\sigma^{n-1}} d\sigma\right) d\tau \\
&= \left[\tau \left(\int_{\tau}^{\infty} \frac{e^{-\sigma}}{\sigma^{n-1}} d\sigma\right)\right] \Big|_{\infty}^{\rho} + \int_{\infty}^{\rho} \frac{e^{-\tau}}{\tau^{n-2}} d\tau \\
&= \rho E_{n-1} - E_{n-2},
\end{aligned}
\tag{21}
$$

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so that $\int_\infty^\rho E_{n-1} d\tau$ can be expressed as the same type of sum. Hence the second term in (19) gives a sum of terms of the form $E_k(2\rho)$ and the third and fourth terms a sum involving $E_k(3\rho)$.

We now carry the process through in the most interesting cases, $n = 2, 3$.

4 The case $k = 0, n = 2$

When $n = 2$ we are interested in

$$E_1(\rho) = \int_\rho^\infty \frac{1}{\tau} e^{-\tau} d\tau$$

$$= -e^{-\rho} \log \rho + \int_\rho^\infty e^{-\tau} \log \tau \, d\tau$$

$$= -e^{-\rho} \log \rho + \int_0^\infty e^{-\tau} \log \tau \, d\tau - \int_0^\rho e^{-\tau} \log \tau \, d\tau$$

$$= -e^{-\rho} \log \rho - \gamma - \rho (\log \rho - 1) e^{-\rho} + O\left(\rho^2 \log \rho\right), \quad \text{for small } \rho,$$

$$= -\log \rho - \gamma + \rho + O\left(\rho^2 \log \rho\right). \tag{22}$$

(See, for example, [2], Chapter 1.) Also, for future purposes, using (20) we obtain

$$E_2(\rho) = \frac{e^{-\rho}}{\rho} - E_1(\rho) \tag{23}$$

$$= \frac{1}{\rho} + \log \rho + (\gamma - 1) - \frac{1}{2}\rho + O\left(\rho^2 \log \rho\right) \text{ as } \rho \to 0. \tag{24}$$

Looking now at (19), with $\rho = \varepsilon$, we see that as $\varepsilon \to 0$,

$$C \log \varepsilon \to -1$$

and

$$C = \frac{1}{\log \frac{1}{\varepsilon}} + O\left(\frac{1}{\left(\log \frac{1}{\varepsilon}\right)^2}\right).$$

Hence the series in (19) is in powers of $\frac{1}{\log \frac{1}{\varepsilon}}$.

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Also, we will work our approximations (in order to compare the results with those of Hinch in [3]) to order $\frac{1}{\log^2\left(\frac{1}{\varepsilon}\right)}$, so that (for example)

$$u = \frac{a\left(r\right)}{\log\left(\frac{1}{\varepsilon}\right)} + \frac{b\left(r\right)}{\log^2\left(\frac{1}{\varepsilon}\right)} + O\left(\log^{-3}\frac{1}{\varepsilon}\right)$$

for any fixed value of r (ρ of order ε). This, as we shall see, necessitates finding

$$C = \frac{1}{\log\left(\frac{1}{\varepsilon}\right)}\left\{1 + \frac{A}{\log\left(\frac{1}{\varepsilon}\right)} + \frac{B}{\log^2\left(\frac{1}{\varepsilon}\right)} + O\left(\log^{-3}\left(\frac{1}{\varepsilon}\right)\right)\right\},$$

and requires use of all the terms in (19).

With this in mind, we look at the second term of (19). Thus from (21),

$$\int_\rho^\infty E_1 d\tau = -\rho E_1 + e^{-\rho}, \tag{25}$$

so that the second term is

$$\begin{aligned}
C^2 \int_\infty^\rho \frac{1}{\tau} e^{-\tau}\left(\tau E_1 - e^{-\tau}\right) d\tau &= C^2\left\{\int_\infty^\rho e^{-\tau} E_1 d\tau - \int_\infty^\rho \frac{e^{-2\tau}}{\tau} d\tau\right\} \\
&= C^2\left\{\left[-e^{-\tau} E_1\right]\big|_\infty^\rho - 2\int_\infty^\rho \frac{e^{-2\tau}}{\tau} d\tau\right\} \\
&= C^2\left(-e^{-\rho} E_1\left(\rho\right) + 2E_1\left(2\rho\right)\right). \tag{26}
\end{aligned}$$

From (22), the second term is therefore

$$\begin{aligned}
C^2 &\left(\log\rho + \gamma - 2\log 2\rho - 2\gamma + O\left(\rho\right)\right) \\
&= C^2\left(-\log\rho - \gamma - 2\log 2 + O\left(\rho\right)\right) \tag{27}
\end{aligned}$$

as $\rho \to 0$.

In the third and fourth terms of (19) we need only the leading terms, i.e. we can ignore the equivalent of $\gamma + 2\log 2$ in (27). Using (25) the third term becomes

$$\frac{1}{2}C^3 \int_\infty^\rho \frac{e^{-\tau}}{\tau}\left(e^{-\tau} - \tau E_1\right)^2 d\tau = \frac{1}{2}C^3\left(\log\rho + O\left(1\right)\right) \text{ as } \rho \to 0. \tag{28}$$

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Finally, in the fourth term, the integrand in the τ-integral is just the second term, (as a function of σ), so that from (25), the fourth term is

$$M = -C^3 \int_\infty^\rho \frac{e^{-\tau}}{\tau} \left[\int_\infty^\tau \left\{ -e^{-\sigma} E_1(\sigma) + 2E_1(2\sigma) \right\} d\sigma \right] d\tau. \tag{29}$$

It is seen from (25) that for any $\tau \le \infty$, $\int_0^\tau E_1(\sigma) d\sigma$ converges. Hence we can write the inner integral above in the form $\int_\infty^0 + \int_0^\tau$, and it follows that

$$M = -C^3 \int_\infty^\rho \frac{e^{-\tau}}{\tau} \left\{ K + r(\tau) \right\} d\tau$$

where K is a constant, r is bounded and $r(\tau) = O(\tau \log \tau)$ as $\tau \to 0$.   It further follows that
$$M = C^3 \left(K E_1(\rho) + O(1) \right) \text{ as } \rho \to 0.$$

We can evaluate K using (25) and (22):

$$\int_0^\infty E_1(2\sigma) d\sigma = \frac{1}{2} \int_0^\infty E_1(u) du = \frac{1}{2},$$

$$\int_0^\infty e^{-\sigma} E_1(\sigma) d\sigma = \left[-\left(e^{-\sigma} - 1\right) E_1 \right]_0^\infty - \int_0^\infty \left(e^{-\sigma} - 1\right) \frac{e^{-\sigma}}{\sigma} d\sigma$$

$$= \lim_{\sigma \to 0} \left\{ -E_1(2\sigma) + E_1(\sigma) \right\} = \lim_{\sigma \to 0} \left(\log 2\sigma - \log \sigma \right) = \log 2. \tag{30}$$

Hence, from (29), the fourth term of (19) is

$$C^3 \left\{ E_1(\rho)(\log 2 - 1) + O(1) \right\} = -C^3 \left\{ (\log 2 - 1) \log \rho + O(1) \right\} \quad \text{as } \rho \to 0. \tag{31}$$

Now setting $\rho = \varepsilon$ and using $(27), (28)$, and (31), we obtain that

$$-1 = -C \left(-\log \varepsilon - \gamma + O(\varepsilon) \right) + C^2 \left(-\log \varepsilon - \gamma - 2\log 2 + O(\varepsilon) \right)$$

$$+ \frac{1}{2} C^3 \left(\log \varepsilon + O(1) \right) - C^3 \left\{ (\log 2 - 1) \log \varepsilon + O(1) \right\}$$

as $\varepsilon \to 0$.   Hence,

$$\frac{1}{\log\left(\frac{1}{\varepsilon}\right)} = C \left(1 - \frac{\gamma}{\log\left(\frac{1}{\varepsilon}\right)} \right) - C^2 \left(1 - \frac{\gamma + 2\log 2}{\log\left(\frac{1}{\varepsilon}\right)} \right) + C^3 \left(\frac{3}{2} - \log 2 \right) + O\left(\log^{-4}\left(\frac{1}{\varepsilon}\right) \right),$$

$$\tag{32}$$

and

$$C = \frac{1}{\log\left(\frac{1}{\varepsilon}\right)} + \frac{A}{\log^2\left(\frac{1}{\varepsilon}\right)} + \frac{B}{\log^3\left(\frac{1}{\varepsilon}\right)} + O\left(\frac{1}{\log^4\left(\frac{1}{\varepsilon}\right)}\right),$$

where

$$-\gamma + A - 1 = 0,$$
$$B - \gamma A - 2A + (\gamma + 2\log 2) + \frac{3}{2} - \log 2 = 0.$$

Hence,

$$A = \gamma + 1$$
$$B = \gamma^2 + 2\gamma + \frac{1}{2} - \log 2.$$

Thus, for fixed r, ρ of order ε, we have, with $\lambda = \log\left(\frac{1}{\varepsilon}\right)$,

$$u - 1 = \left(\frac{1}{\lambda} + \frac{\gamma + 1}{\lambda^2} + \frac{(\gamma + 1)^2 - \frac{1}{2} - \log 2}{\lambda^3}\right)(\log r + \log \varepsilon + \gamma)$$
$$+ \frac{1}{\lambda^2}\left(1 + \frac{2(\gamma + 1)}{\lambda}\right)(-\log r - \log \varepsilon - \gamma - 2\log 2)$$
$$+ \frac{1}{\lambda^3}\left(\frac{3}{2} - \log 2\right)(\log r + \log \varepsilon) + O\left(\lambda^{-4}\right),$$

so that, after cancellation,

$$u = \frac{\log r}{\lambda} + \frac{\gamma \log r}{\lambda^2} + O\left(\lambda^{-3}\right).$$

This is the "inner expansion". For the "outer expansion", i.e. fixed ρ, r of order $\frac{1}{\varepsilon}$, we use (19), truncated to second order, to get

$$u - 1 = -E_1(\rho)\left(\frac{1}{\lambda} + \frac{\gamma + 1}{\lambda^2}\right) + \frac{1}{\lambda^2}\left(2E_1(2\rho) - e^{-\rho}E_1(\rho)\right) + O\left(\lambda^{-3}\right).$$

These results are in accordance with those of Hinch and of others on this problem.

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5   The case $k = 0, n = 3$

Here we are interested in (from (22) and (23))

$$E_2 \left(\rho \right) = \frac{e^{-\rho}}{\rho} - E_1 \left(\rho \right) = \frac{1}{\rho} + \log \rho + (\gamma - 1) - \frac{1}{2} \rho + O \left(\rho^2 \log \rho \right) \text{ as } \rho \to 0.$$

Thus, the first term on the right of (19) evaluated at $\rho = \varepsilon$ is

$$-C \left(1 + \varepsilon \log \varepsilon + (\gamma - 1) + O \left(\varepsilon^2 \right) \right) \text{ as } \varepsilon \to 0.$$

The second term is

$$(C\varepsilon)^2 \int_\infty^\rho \frac{1}{\tau^2} e^{-\tau} \left(\int_\infty^\tau E_2 d\sigma \right) d\tau$$

$$= (C\varepsilon)^2 \left\{ \left[-E_2 \left(\tau \right) \int_\infty^\tau E_2 \left(\sigma \right) d\sigma \right]_\infty^\rho + \int_\infty^\rho E_2^2 d\tau \right\}$$

$$= (C\varepsilon)^2 \left\{ -E_2 \left(\rho \right) \int_\infty^\rho E_2 \left(\tau \right) d\tau + \int_\infty^\rho E_2^2 d\tau \right\}.$$

From (23) we see that

$$\int_\infty^\rho E_2^2 d\tau = -\frac{1}{\rho} + \log^2 \rho + O \left(\log \rho \right) \text{ as } \rho \to 0,$$

while from (21),

$$\int_\infty^\rho E_2 d\tau = \rho E_2 - E_1 = 1 + \log \rho + \gamma + O \left(\rho \log \rho \right),$$

$$E_2 \int_\infty^\rho E_2 d\tau = \frac{1}{\rho} \log \rho + \frac{\gamma + 1}{\rho} + O \left(\log^2 \rho \right).$$

In all, the second term is

$$(C\varepsilon)^2 \left\{ -\frac{1}{\rho} \log \rho - \frac{\gamma + 2}{\rho} + O \left(\log^2 \rho \right) \right\}.$$

It is readily verified that the third and fourth terms in (19) give $O \left\{ C^3 \varepsilon^3 \left(\frac{1}{\rho} \log^2 \rho \right) \right\}$, which is negligible. Thus, evaluating (19) at $\rho = \varepsilon$, we have

$$-1 = -C\varepsilon \left(\frac{1}{\varepsilon} + \log \varepsilon + \gamma - 1 \right) + (C\varepsilon)^2 \left(-\frac{1}{\varepsilon} \log \varepsilon - \frac{\gamma + 2}{\varepsilon} \right) + O \left(C^3 \varepsilon^2 \log^2 \varepsilon \right),$$

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so that
$$C = 1 - 2\varepsilon \log \varepsilon - \varepsilon (2\gamma + 1) + O\left(\varepsilon^2 \log^2 \varepsilon\right).$$

Then, for fixed r, ρ of order ε, we have

$$u - 1 = -\varepsilon \left(1 - 2\varepsilon \log \varepsilon - \varepsilon (2\gamma + 1)\right) \left(\frac{1}{\varepsilon r} + \log \varepsilon + \log r + \gamma - 1\right)$$
$$+ \varepsilon^2 \left(-\frac{1}{\varepsilon r} (\log \varepsilon + \log r) - \frac{\gamma + 2}{\varepsilon r}\right) + O\left(\varepsilon^2 \log^2 \varepsilon\right),$$

$$u = 1 - \frac{1}{r} - \varepsilon \log \varepsilon \left(1 - \frac{1}{r}\right) - \varepsilon \left(\log r + \frac{\log r}{r}\right) + \varepsilon (1 - \gamma) \left(1 - \frac{1}{r}\right)$$
$$+ O\left(\varepsilon^2 \log^2 \varepsilon\right).$$

For fixed ρ, r of order ε^{-1}, we again use (19), to give

$$u - 1 = -\varepsilon \left(1 - 2\varepsilon \log \varepsilon - \varepsilon (2\gamma + 1)\right) E_2 (\rho)$$
$$+ \varepsilon^2 \left\{E_1 (\rho) E_2 (\rho) - \rho E_2^2 (\rho) - \int_\rho^\infty E_2^2 d\tau\right\} + O\left(\varepsilon^3\right). \tag{33}$$

Again, these results are in agreement with those of Hinch, and others, although (33) gives one term further.

Remark 3. It is of interest to consider what happens when $n < 2$, since, at least for $n \geq 1$, there still exists a unique solution. The equation (19) is still valid at $\rho = \varepsilon$, but since $E_{n-1} (\rho)$ is no longer singular at $\rho = 0$ for $n < 2$, (19) with $\rho = \varepsilon$ becomes merely an implicit equation for $C\varepsilon^{n-2}$. This tells us that $C \to 0$, since $\varepsilon^{n-2} \to \infty$, but we no longer get an asymptotic expansion. In particular, it is no longer obvious that C is unique. Of course, we know this from Theorem 1 if $n \geq 1$.

6   The case $k = 1$

We can in fact treat a generalization which causes no further difficulties,

$$u'' + \frac{n-1}{r} u' + f(u) u'^2 + \varepsilon u \, u' = 0, \tag{34}$$

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with the same boundary conditions. As before, we will compare our results with those of Hinch in [3].

As remarked in the proof of Theorem 1, the solution will necessarily have $u' > 0$ so that conditions on $f(u)$ are necessary only for $0 \le u \le 1$. We require only that f be continuous and positive in this interval.

Then (34) can be written as

$$\frac{(r^{n-1}u')'}{r^{n-1}u'} + f(u)u' + \varepsilon u = 0,$$

so that

$$\log\left(r^{n-1}u'\right) = -F(u) - \varepsilon \int_1^r u\, dt + A$$

for some constant A, where

$$F(u) = \int_0^u f(s)\, ds.$$

This becomes

$$e^{F(u)}u' = \frac{C}{r^{n-1}}e^{-\varepsilon r - \varepsilon \int_\infty^r (u-1)ds},$$

or, on integration,

$$G(u) - G(1) = C \int_\infty^r \frac{1}{t^{n-1}}e^{-\varepsilon t - \varepsilon \int_\infty^t (u-1)ds}\, dt,$$

where

$$G(u) = \int_0^u e^{F(v)}\, dv.$$

In order to keep the manipulations simple and effect comparisons, we will consider from here the Lagerstrom model, where $f(u) = 1$, $F(u) = u$, $G(u) = e^u - 1$. Then, with $\varepsilon r = \rho$, $\varepsilon t = \tau$, we have

$$e^u - e = C\varepsilon^{n-2} \int_\infty^\rho \frac{1}{\tau^{n-1}}e^{-\tau}e^{-\int_\infty^\tau (u-1)d\sigma}\, d\tau, \tag{35}$$

and writing

$$u - 1 = \frac{u-1}{e^u - e}\left(e^u - e\right),$$

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we get

$$e^u - e = C\varepsilon^{n-2} \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau} e^{-\int_\infty^\tau \frac{u-1}{e^u-e}(e^u-e)d\sigma} d\tau. \tag{36}$$

As in section 3, we can integrate by parts, and since $0 \le \frac{u-1}{e^u-e} \le 1$ in $0 \le u < 1$, we will develop a convergent series as before. To get the first three terms (necessary to give Hinch's accuracy when $n = 2$), we have from (36) that

$$e^u - e$$

$$= C\varepsilon^{n-2} \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau} \left\{ 1 - \int_\infty^\tau \frac{u-1}{e^u-e}(e^u-e)\,d\sigma + \frac{1}{2}\left(\int_\infty^\tau \frac{u-1}{e^u-e}(e^u-e)\,d\sigma\right)^2 + \cdots \right\} d\tau. \tag{37}$$

As before, since $e^u - e \to 0$ exponentially fast as $\rho \to \infty$, the series in the integrand converges uniformly for large τ, so that (37) is valid for large ρ. But again we need to extend it down to $\rho = \varepsilon$. From (35) we have

$$e^u - e \le C\varepsilon^{n-2} E_{n-1}(\rho)$$

and so the convergence proof is the same as that preceding (19).

Before proceeding further with $n = 2$, we make a couple of remarks about the simpler case $n > 2$. Then, as we saw in subsection 5, only two terms are necessary to give the required accuracy, and then (37) gives

$$e^u - u = C\varepsilon^{n-2} \int_\infty^\rho \frac{e^{-\tau}}{\tau^{n-1}} \left\{ 1 - \int_\infty^\tau \frac{u-1}{e^u-e}(e^u-e)\,d\sigma + \cdots \right\}$$

and since $\frac{u-1}{e^u-e}$ appears in what is already the highest order term, we can replace it by its limit as $u \to 1$, i.e. $\frac{1}{e}$. Thus we get, to the required order,

$$e^u - e = -C\varepsilon^{n-2} E_{n-1} - \left(\frac{C\varepsilon^{n-2}}{e}\right) \int_\infty^\rho \frac{e^{-\tau}}{\tau^{n-1}} \int_\infty^\tau (e^u-e)\,d\sigma d\tau.$$

(We will proceed more carefully for $n = 2$.) This, apart from the factor $\frac{1}{e}$, is the same equation as we dealt with in section 5 (with $e^u - e$ in place of $u - 1$), and the solution can be written down from there. (If we had a general function f in place of 1, we would get

$$e^{F(u)} - e^{F(1)} = -C\varepsilon^{n-2} E_{n-1} - \frac{C\varepsilon^{n-2}}{e^{F(1)}f(1)} \int_\infty^\rho \frac{e^{-\tau}}{\tau^{n-1}} \left(\int_\infty^\tau \left(e^{F(u)} - e^{F(1)}\right) d\sigma\right) d\tau.)$$

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Turning now to the case $n = 2$, and $F(u) = u$, we need three terms on the right of (37). Thus,

$$\frac{u-1}{e^u - e} = \frac{1}{e} - \frac{1}{2e^2}(e^u - e) + O(e^u - e)^2 \quad \text{as } u \to 1. \tag{38}$$

We follow the method used just before (19) and obtain from (37) that

$$e^u - e = -C\varepsilon^{n-2}E_{n-1} + \frac{1}{e}\left(C\varepsilon^{n-2}\right)^2 \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau} \left(\int_\infty^\tau E_{n-1}d\sigma\right) d\tau$$

$$+ \frac{1}{2e^2}\left(C\varepsilon^{n-2}\right)^3 \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau} \left(\int_\infty^\tau E_{n-1}^2 d\sigma\right) d\tau$$

$$+ \frac{1}{2e^2}\left(C\varepsilon^{n-2}\right)^3 \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau} \left(\int_\infty^\tau E_{n-1}d\sigma\right)^2 d\tau$$

$$- \frac{1}{e^2}\left(C\varepsilon^{n-2}\right)^3 \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau} \int_\infty^\tau \left\{\int_\infty^\sigma \frac{1}{s^{n-1}} e^{-s} \left(\int_\infty^s E_{n-1}dt\right) ds\right\} d\sigma d\tau + O\left(\Phi^4\right)$$

$$= -C\varepsilon^{n-2}E_{n-1} + F_1 + F_2 + F_3 + F_4 + O\left(\Phi^4\right), \quad \text{say.}$$

As before, if $n > 2$ then this is valid for any C as $\varepsilon \to 0$, uniformly in $\rho \geq \varepsilon$, while if $n = 2$, it is valid as $C \to 0$.

For $n = 2$ we can continue to follow the argument in section 4. Thus, as $\rho \to 0$,

$$F_1 = \frac{1}{e}C^2\left(-\log\rho - \gamma - 2\log 2 + O(\rho)\right)$$

$$F_3 = \frac{1}{2e^2}C^3\left(\log\rho + O(1)\right)$$

$$F_4 = -\frac{1}{e^2}C^3\left[(\log 2 - 1)\log\rho + O(1)\right].$$

The term F_2 did not appear before. Only the highest order term is needed for our expansion and this is

$$-\frac{1}{2e^2}C^3\left(\int_\infty^\rho \frac{1}{\tau} e^{-\tau} d\tau\right) \int_0^\infty E_1^2 d\sigma.$$

Now

$$\int_0^\infty E_1^2 d\sigma = \left[\tau E_1^2\right]_0^\infty + 2\int_0^\infty \tau \frac{e^{-\tau}}{\tau} E_1 d\tau$$

$$= 2\int_0^\infty e^{-\tau} E_1 d\tau = 2\log 2, \quad \text{from } (30).$$

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Thus,

$$F_2 = \frac{1}{e^2}C^3 E_1 \left(\log 2 + O\left(\rho \log^2 \rho\right)\right) = -\frac{1}{e^2}C^3 \left((\log 2)\log \rho + O\left(1\right)\right),$$

and, evaluating at $\rho = \varepsilon$, we have

$$1 - e = C\left(\log \varepsilon + \gamma + O\left(\varepsilon\right)\right)$$
$$-\frac{1}{e}C^2 \left(\log \varepsilon + \gamma + 2\log 2 + O\left(\varepsilon\right)\right) + \frac{1}{2e^2}C^3 \log \varepsilon \left(-2\log 2 + 1 - 2\log 2 + 2\right) + O\left(C^3\right),$$

$$\frac{e-1}{\log \frac{1}{\varepsilon}} = C\left(1 - \frac{\gamma}{\log\left(\frac{1}{\varepsilon}\right)}\right) - \frac{1}{e}C^2 \left(1 - \frac{\gamma + 2\log 2}{\log\left(\frac{1}{\varepsilon}\right)}\right)$$
$$+ \frac{1}{2e^2}C^3 \left(3 - 4\log 2 + O\left(\frac{C\varepsilon}{\log \varepsilon}\right) + O\left(\frac{C^2\varepsilon}{\log \varepsilon}\right) + O\left(\frac{C^3}{\log \varepsilon}\right)\right).$$

Hence if

$$C = \frac{e-1}{\log\left(\frac{1}{\varepsilon}\right)} + \frac{A}{\log^2\left(\frac{1}{\varepsilon}\right)} + \frac{B}{\log^3\left(\frac{1}{\varepsilon}\right)} + O\left(\log^{-4}\left(\frac{1}{\varepsilon}\right)\right),$$

then

$$-\gamma\left(e-1\right) + A - \frac{\left(e-1\right)^2}{e} = 0,$$

$$A = \frac{e-1}{e}\left(\gamma e + e - 1\right),$$

$$B - A\gamma + \frac{\left(e-1\right)^2}{e}\left(\gamma + 2\log 2\right) - \frac{2A\left(e-1\right)}{e} + \frac{1}{2e^2}\left(e-1\right)^3\left(3 - 4\log 2\right) = 0.$$

We can of course calculate B, but in fact its value will be be irrelevant to the level of approximation that we take.

Then, for fixed r (ρ of order ε), we have, with $l = \log\left(\frac{1}{\varepsilon}\right)$,

$$e^u - e = \left(e-1\right)\left\{\frac{1}{l} + \frac{\gamma + 1 - \frac{1}{e}}{l^2} + \frac{B/\left(e-1\right)}{l^3}\right\}\left(\log \varepsilon + \log r + \gamma\right)$$
$$+ \frac{1}{e}\left(e-1\right)^2 \left\{\frac{1}{l^2} + \frac{2\left(\gamma + 1 - \frac{1}{e}\right)}{l^3}\right\}\left(-\log \varepsilon - \log r - \gamma - 2\log 2\right)$$
$$+ \frac{1}{2e^2}\frac{\left(e-1\right)^3}{l^3}\left(3 - 4\log 2\right)\left(\log \varepsilon + \log r\right) + O\left(l^{-3}\right)$$
$$= 1 - e + \frac{\left(e-1\right)\log r}{l} + \frac{\gamma\left(e-1\right)\log r}{l^2} + O\left(l^{-3}\right). \tag{39}$$

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(Note that the definitions of A and B were such that $u = 0$ at $r = 1$ up to and including order l^{-2}, so that to that order there can be only terms in $\log r$, not constant terms. We do not need the explicit value of B.) To obtain u, we have to invert, so that

$$u = \log \left\{ 1 + \frac{e-1}{l} \log r + \frac{\gamma(e-1)}{l^2} \log r + O\left(l^{-3}\right) \right\}.$$

For fixed ρ, r of order ε^{-1}, we have

$$e^u - e = -\frac{e-1}{l}\left(1 + \frac{\gamma + 1 - \frac{1}{e}}{l}\right) E_1(\rho) + \frac{(e-1)^2}{el^2}\left(2E_1(2\rho) - e^{-\rho}E_1(\rho)\right) + O\left(l^{-3}\right).$$

Thus

$$
\begin{aligned}
u - 1 &= \frac{1}{e}\left(e^u - e\right) - \frac{1}{2e^2}\left(e^u - e\right)^2 + \cdots \\
&= -\frac{e-1}{e}\left(1 + \frac{\gamma + 1 - \frac{1}{e}}{l}\right)\frac{E_1(\rho)}{l} + \frac{(e-1)^2}{e^2}\frac{\left(2E_1(2\rho) - e^{-\rho}E_1(\rho)\right)}{l^2} \\
&\quad - \frac{(e-1)^2}{2e^2}\frac{E_1^2(\rho)}{l^2} + O\left(l^{-3}\right).
\end{aligned}
\tag{40}
$$

Again, these results are consistent with those of Hinch and others, except that Hinch has an algebraic mistake which in (40) replaces $\gamma + 1 - \frac{1}{e}$ by $\gamma - 1 + \frac{1}{e}$.

7 Final Remarks

Starting with Lagerstrom, the terms involving $\log \varepsilon$ in the inner expansions have been considered difficult to explain. They are often called "switchback" terms, because there is nothing obvious in the equation which indicates the need for such terms, and because, starting with an expansion in powers of ε, one finds inconsistent results which are only resolved by adding terms of lower order, that is, powers of $\varepsilon \log \varepsilon$. The recent approach to the problem by geometric perturbation theory explains this by reference to a "resonance phenomenon", which is too complicated for us to describe here [10],[11].

In our work, the necessity for such terms is seen already from the equation (11) and the resulting expansion (17) :

$$u(\rho) - 1 = C\varepsilon^{n-2} \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau} \left\{ 1 - \int_\infty^\tau (u-1)\, d\sigma + \frac{1}{2}\left(\int_\infty^\tau (u-1)\, d\sigma\right)^2 - \cdots \right\} d\tau.$$

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In the existence proof it was seen in (9) that $C = O(1)$ as $\varepsilon \to 0$. On the right of (17) the first term is simply $-C\varepsilon^{n-2}E_{n-1}(\rho)$, and the simple expansions given for E_1 and E_2 show immediately the need for the logarithmic terms. There is no "switchback", because the procedure does not start with any assumption about the nature of the expansion, and there is no need for a "matching".

A number of authors have noted that the outer expansion is a uniformly valid asymptotic expansion on $[1, \infty)$, and therefore it "contains" the inner expansion [5], though this is more subtle when $k = 1$ [13]. Our twist on this is that both expansions are contained in the uniformly convergent series defined implicitly by (17). The simple derivation of this series, via the integral equation (11) is new, as far as we know.

References

[1] Cohen, D. S., A. Fokas and P. Lagerstrom, Proof of some asymptotic results for a model equation for low Reynolds number flow, SIAM J. Appld. Math. 35 (1978), 187-207.

[2] Erdelyi, A. et. al., Higher Transcendental Functions, Vol. 1, McGraw Hill, 1953.

[3] Hinch, E. J., Perturbation Methods, Cambridge University Press, 1991.

[4] Hsiao, G. C., Singular perturbations for a nonlinear differential equation with a small parameter, Siam J. Math Anal. 4 (1973), 283-301.

[5] Hunter, C., M. Tajdari and S. D. Boyer, On Lagerstrom's model of slow incompressible viscous flow, Siam J. Appld. Math.50 1990, 48-63.

[6] Kaplun, S. and P. A. Lagerstrom, Asymptotic expansions of Navier-Stokes solutions for small Reynolds number, J. Math. Mech. 6 (1957), 585-593.

[7] Lagerstrom, P. A. and R. G. Casten, Basic Concepts in Singular Perturbation Techniques, Siam Review 14 (1972), 63-120.

[8] Lagerstrom, P.A. and Reinelt, C. A., Note on logarithmic and switchback terms in regular and singular perturbation expansions, Siam Review 44 (1984), 451-462.

[9] MacGillivray, A. D., On a model equation of Lagerstrom, Siam J. Appld. Math. 34 (1978), 804-812.

[10] Popovic, N. and P. Szmolyan, A geometric analysis of the Lagerstrom model problem, J. Differential Equations 199 (2004), 290-325.

[11] Popovic, N. and P. Szmolyan, Rigorous asymptotic expansions for Lagerstrom's model equations, a geometric approach, Nonlinear Analysis 59 (2004), 531-565.

[12] Rosenblat, S. and J. Shepherd, On the asymptotic solution of the Lagerstrom model equation, Siam J. Appld. Math. 29 (1975), 110-120.

[13] Skinner, L. A., Note on the Lagerstrom singular perturbation models, Siam J. Appld. Math. 41 (1981), 362-364.

[14] Tam, K. , On the Lagerstrom model for flow at low Reynolds numbers, J. Math. Anal. Appl.., 49 (1975), 286-294.